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1982 J. Phys. A: Math. Gen. 15 L395

(http://iopscience.iop.org/0305-4470/15/8/004)

J. Phys. A: Math. Gen. **15** (1982) L395–L398. Printed in Great Britain

LETTER TO THE EDITOR

Random graphs and network communication

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Received 26 April 1982

Abstract. The problem of random graphs, which arises in the analysis of network reliability in communication theory, is considered here as a bond percolation. A closed form expression is obtained for the cluster-size generating function from which the mean cluster size as well as the percolation probability are derived. In a network of $N \to \infty$ stations in which the communication between any two stations is intact with a probability α/N, it is found that for $\alpha \leq 1$ the network breaks into clusters of average size of $(1-\alpha)^{-1}$ stations and $\alpha/(1-\alpha)$ links, while for $\alpha > 1$ there is a non-zero percolation probability.

A random graph is a collection of N vertices (sites) which are governed by a probability mechanism such that each pair of vertices is joined by an edge with a prescribed probability p, independent of the presence or absence of any other edges. If we regard the vertices as stations and the edges as communication links between the stations, then the random graphs simulate a communication network (see e.g. Welsh 1977). Writing $p = \alpha/N$ and α small, we expect the network to break down, even in the limit of $N \to \infty$, and decompose into isolated clusters of finite sizes which are not linked to one another via communication. But for α greater than a certain critical value α_c, a non-zero probability arises that a given station is linked with an infinite number of other stations.

The random graphs so defined also describe a bond percolation process (Welsh 1977), if the edges are regarded as occupying bonds. This consideration provides the possibility of an alternative approach to the problem of network reliability, a possibility which appears not to have been adequately examined. In this Letter we take up this consideration. We shall first formulate the percolation problem as a Potts model (Kasteleyn and Fortuin 1969), which is soluble in the limit of $N \to \infty$. Relevant information regarding the network reliability and random graphs is then derived from this solution.

We begin by writing down the Potts Hamiltonian relevant to the percolation problem. Since the bond percolation is long ranged in the sense that any two vertices can be connected, we consider a system of q-state Potts spins (for a review on the Potts model see Wu (1982)) having a similar long-range interaction. Thus, we consider the Hamiltonian $\mathcal{H}$ given by

$$\frac{-\mathcal{H}}{kT} = \frac{K}{N} \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j) + \frac{M}{N} \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j)\delta(\sigma_i, 0) + L \sum_i \delta(\sigma_i, 0), \tag{1}$$

where, in addition to the two-spin interactions K/N between *all* pairs $\langle ij \rangle$, there are also external fields M/N and L (cf equation (1.18) of Wu (1982)). These external fields are needed to generate quantities relevant to the cluster size, and will eventually

0305-4470/82/080395 + 04$02.00

be set to zero. In (1), $\sigma_i = 0, 1, \ldots, q-1$ refers to the spin state at the ith site, $i = 1, 2, \ldots, N$, and $\delta(\alpha, \beta)$ is the Kronecker delta function.

Consider the random graphs now regarded as configurations of the bond percolation. Following Kasteleyn and Fortuin (1969), we can establish that this bond percolation is generated by the Potts model (1). More specifically (cf equation (4.9) of Wu (1982)), let $Z(q; K, M, L)$ be the partition function of (1) and write

$$A(q; K, M, L) = (1/N) \ln Z(q; K, M, L); \tag{2}$$

then the cluster-size generating function, $G(L, L_1)$ for the percolation process is given by

$$G(L, L_1) \equiv \frac{1}{N}\left\langle \sum_c \exp(-Ls_c - L_1 b_c)\right\rangle = \left(\frac{\partial}{\partial q} A(q; K, M, L)\right)_{q=1} \tag{3}$$

with

$$e^{L_1} = (e^{(K+M)/N} - 1)/(e^{K/N} - 1). \tag{4}$$

Here the average $\langle\ \rangle$ is taken over all bond percolation configurations with the bond occupation probability

$$\alpha/N = 1 - e^{-(K+M)/N}, \tag{5}$$

the summation Σ_c in (3) is taken over all clusters of the percolation configuration and s_c, b_c denote, respectively, the numbers of sites and bonds in a cluster.

The cluster-size generating function $G(L, L_1)$ generates the various quantities of interest in the percolation problem. In particular, the percolation probability $P(\alpha)$ and the mean cluster size $S(\alpha)$ (of the finite cluster containing a given vertex) are given by (see e.g. Wu 1978)

$$P(\alpha) = 1 + M_{10}(\alpha), \tag{6}$$

$$\begin{aligned} S(\alpha) &= M_{20}(\alpha) \qquad \text{by site content,} \\ &= M_{11}(\alpha) \qquad \text{by bond content,} \end{aligned} \tag{7}$$

where

$$M_{rs}(\alpha) = \left(\frac{\partial^{r+s}}{\partial L^r \partial L_1^s} G(L, L_1)\right)_{L=L_1=0+}. \tag{8}$$

We next proceed to compute $G(L, L_1)$ by solving the Potts model (1). For N large, (4) and (5) give

$$K = \alpha\, e^{-L_1}, \qquad M = \alpha(1 - e^{-L_1}). \tag{9}$$

Also, in the limit of $N \to \infty$, Hamiltonian (1) is most conveniently dealt with by using a variational approach (Wu 1982).

Let x_i denote the fraction of spins that are in the spin state $i = 0, 1, \ldots, q-1$. We look for a solution with a long-range order in, say, the $i = 0$ spin state. To this end we write

$$x_0 = (1/q)[1 + (q-1)s], \qquad x_i = (1/q)(1-s), \qquad i \neq 0, \tag{10}$$

where $0 \leqslant s \leqslant 1$ is the order parameter. We then obtain from (1) and (2):

$$A(q;K,M,L)=\max_{0\leqslant s\leqslant 1}\left[\frac{K}{2q}[1+(q-1)s^2]+\frac{Mu^2}{2q^2}+\frac{Lu}{q}+\ln q-u\ln u\right.$$
$$\left.-\left(\frac{q-1}{q}\right)(1-s)\ln(1-s)\right] \tag{11}$$

where $u=1+(q-1)s$. Let s_0 be the value of the order parameter which maximises (11). Straightforward algebra leads to

$$A(q;K,M,L)=\frac{K}{2q}[(1-s_0)^2-qs_0^2]-\frac{M}{2}\left(s_0+\frac{1}{q}(1-s_0)\right)^2-\ln\left(\frac{1-s_0}{q}\right) \tag{12}$$

where s_0 is determined from

$$Ks_0+L+\frac{M}{q}[1+(q-1)s_0]=\ln\left(\frac{1+(q-1)s_0}{1-s_0}\right). \tag{13}$$

Substitution of (12) into (3) after using (9) now yields

$$G(L,L_1)=1-s_0-\tfrac{1}{2}\alpha\ \mathrm{e}^{-L_1}(1-s_0)^2 \tag{14}$$

where s_0 is determined from (13) at $q=1$, which now becomes, after introducing (9),

$$\alpha-\alpha\ \mathrm{e}^{-L_1}(1-s_0)+L+\ln(1-s_0)=0. \tag{15}$$

Finally, we obtain from (6), (7), and (14) and (15) the results

$$P(\alpha)=s_0, \tag{16}$$

$$S(\alpha)=(1-s_0)/(1-\alpha+\alpha s_0) \qquad \text{by site content,}$$
$$=\alpha/(1-\alpha+\alpha s_0) \qquad \text{by bond content,} \tag{17}$$

where s_0 is determined from

$$\alpha s_0+\ln(1-s_0)=0. \tag{18}$$

Equations (16)–(18) are our main results. For $\alpha \leqslant \alpha_c = 1$, (18) has only one solution, namely, $s_0=0$, so that $P(\alpha)=0$ identically; the mean cluster size is then $(1-\alpha)^{-1}$ by site content and $\alpha/(1-\alpha)$ by bond content. For $\alpha>\alpha_c$, another solution $s_0>0$ arises which gives rise to a larger A (as seen from (11) in the limit of $q\to 1+$). Therefore, we should take this solution, and this leads to a non-zero percolation probability $P(\alpha)=s_0$. Near the threshold $\alpha_c=1$, (16), (17) and (18) give

$$P(\alpha)=2(\alpha-\alpha_c), \qquad S(\alpha)=|\alpha-\alpha_c|^{-1}, \tag{19}$$

leading to the classical percolation exponents $\beta=\gamma=\gamma'=1$. It is not surprising that we should obtain these 'mean field' exponents, since the expression (1) describes precisely a mean field Hamiltonian (Kac 1968) for the Potts model.

Erdös and Rényi (1960) have studied an equivalent graph problem in which the number of connecting edges is fixed at $\alpha N/2$, the average number of edges in the present problem. Using a purely probabilistic approach, they showed that the cluster structures of the random graphs exhibit a drastic change at $\alpha=1$. Therefore, our results are consistent with their findings. After the completion of this work I learned that A Coniglio has also considered this percolation problem using a different variational approach.

To summarise, we have considered the problem of network reliability using a Potts model approach. In a network of $N \to \infty$ stations, where the communication between any two stations is intact with a probability α/N, we have found that, for $\alpha \leq 1$, the network breaks into clusters of stations which have an average size of $(1-\alpha)^{-1}$ stations and $\alpha/(1-\alpha)$ communication lines. When $\alpha > 1$, there is a non-zero probability $P(\alpha)$, obtained from $\alpha P(\alpha) + \ln[1 - P(\alpha)] = 0$, that a given station maintains communication with an infinite number of other stations.

I wish to thank H E Stanley for the kind hospitality at the Center for Polymer Physics, Boston University, where this work was initiated. This research has been supported in part by grants from the National Science Foundation and the US Army Research Office.

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